

A Stackelberg Solution to a Two-Level Linear Fractional Programming Problem with Interval Coefficients in the Objective Functions

(Penyelesaian Stackelberg bagi Masalah Pengaturcaraan Pecahan Linear Dua-Aras dengan Pekali Selang dalam Fungsi Objektif)

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ABSTRACT

In this paper, two approaches were introduced to obtain Stackelberg solutions for two-level linear fractional programming problems with interval coefficients in the objective functions. The approaches were based on the K th best method and the method for solving linear fractional programming problems with interval coefficients in the objective function. In the first approach, linear fractional programming with interval coefficients in the objective function and linear programming were utilized to obtain Stackelberg solution, but in the second approach only linear programming is used. Since a linear fractional programming with interval coefficients can be equivalently transformed into a linear programming, therefore both of approaches have same results. Numerical examples demonstrate the feasibility and effectiveness of the methods.

Keywords: Interval coefficients; linear fractional programming; Stackelberg solution; two-level programming

ABSTRAK

Dalam kajian ini, dua kaedah diperkenalkan untuk mendapatkan penyelesaian Stackelberg bagi masalah pengaturcaraan pecahan linear dua-aras dengan pekali selang dalam fungsi objektif. Kaedah yang digunakan adalah berdasarkan kaedah terbaik peringkat- K dan kaedah penyelesaian masalah pengaturcaraan pecahan linear dengan pekali selang dalam fungsi objektif. Dalam kaedah pertama, pengaturcaraan pecahan linear dengan pekali selang dalam fungsi objektif dan pengaturcaraan linear digunakan untuk mendapatkan penyelesaian Stackelberg, tetapi dalam kaedah kedua hanya pengaturcaraan linear digunakan. Oleh sebab suatu pengaturcaraan pecahan linear dengan pekali selang boleh dijemakan secara setara kepada pengaturcaraan linear, kedua-dua kaedah menghasilkan keputusan yang sama. Beberapa contoh berangka menunjukkan kesauran dan keberkesanan kaedah-kaedah ini.

Kata kunci: Pekali selang; pengaturcaraan dua-aras; pengaturcaraan pecahan linear; penyelesaian Stackelberg

INTRODUCTION

The two-level mathematical programming problems are used to model decision making problems in the real world of decentralized organizations (Anandalingam & Friesz 1992; Bard & Moore 1990; Bard 1998; Luo et al. 1996; Migdallas et al. 1998; Stackelberg 1934). In the two-level programming problem two decision makers make a decision successively. The decision maker at the upper level (leader) specifies a strategy then the decision maker at the lower level (follower) specifies a strategy to optimize the objective function of self with full knowledge of leader's action. Finally the leader optimizes his objective function according to the rational response of the follower. The obtained solution described in the above situation is called a Stackelberg solution (Sakawa & Nishizaki 2009). The three categories for obtaining Stackelberg solutions to two-level linear programming problems are considered: the vertex enumeration (Bialas & Karwan 1984), the Khun-Tucker approach (Bard & Falk 1982; Bard & Moore 1990; Bialas & Karwan 1984; Hansen et al. 1992) and penalty function approach (Annadalingam & White 1990; White

& Annadalingam 1993). The vertex enumeration approach takes advantage of a property that the Stackelberg solution belongs to the extreme point's family of the feasible region.

The K th best method proposed by Bialas and Karwan (1984) can be thought of as the vertex enumeration approach. The solution search procedure of the method starts out from a point which is an optimal solution to the problem of the leader and checks for an optimal solution to the follower. If the first point is not the Stackelberg solution, the procedure continues to examine the second best solution to the problem of the leader and so forth (Sakawa & Nishizaki 2009).

Extending the two-level linear programming, some algorithms were proposed to obtain a Stackelberg solution to three-level linear programming problems (Bard 1984; Wen & Bialas 1986). The concept of the Stackelberg solution arises when there is no cooperative relation between the decision makers. This point must be mentioned that, in many real world decision problems, there is a cooperative relation between decision makers

and consequently the related two-level mathematical programming problems must be considered cooperatively and also some approaches have been proposed to deal with these kind of problems (Sakawa et al. 2000a, 2000b; Shih et al. 1996).

The fractional programming (FP) is a special case of a nonlinear programming, which is generally used for modeling real life problems with one or more objectives such as profit/cost, actual cost/standard and output/employee. The FP is applied to different disciplines such as engineering, business, finance and economics.

The linear fractional programming (LFP) is a special class of FP which can be transformed into a linear programming problem by the method of Charnes and Cooper (1962). The same problem can also be solved by adopting the updated objective function method as discussed by Bitran and Novaes (1973). Stancu-Minasian (1997) gives a survey of FP covering applications as well as major theoretical and algorithmic developments.

The work by Sakawa and Nishizaki (2001) is based on interactive fuzzy programming. They solved a two-level LFP problem when there is a cooperative relation between decision makers. Bearing in mind, they described the *Kth* best method to the two-level programming problem when there is a non-cooperative relation between decision makers based on variable transformation by Charnes and Cooper (1962). As we know, there are many phenomena in the real physical world in which the coefficients are not certain when they are modeled mathematically. Therefore, in such cases it is much better to select coefficients as intervals instead of fixed numbers.

Under these circumstances, in this paper, the two-level LFP problems with interval parameters in objective functions were considered for the case of no cooperative relation between decision makers. After showing that an LFP problem with interval coefficients in the objective function can be equivalently transformed into an LP, two different approaches based on *Kth* method were introduced to obtain Stackelberg solution. In the first approach, LFP with interval coefficients and related LP played a role to obtain Stackelberg solution, but in the second approach only LP was used.

METHOD

When we deal with minimization of an LFP problem with interval coefficients in the objective function under some constraints, two different cases are considered: the numerator is positive for all feasible points and consequently, in this situation the best solution is achieved when the last point of intervals in the denominator is selected. Otherwise, the first point of intervals in denominator must be used. Since it is not possible for us to realize that for all feasible points of a feasible region the numerator is positive, two different LFP problems must be considered simultaneously. Then by comparing the best solution of the two problems, the optimal solution is

computed. Fortunately, the LFP with interval coefficients in the objective function has been solved by transforming it into a linear programming problem (Borza et al. 2012).

The method used in this paper is based on the *Kth* best method and the proposed method by Borza et al. (2012) as given:

$$\text{Minimize } \frac{[a_1, b_1]x_1 + \dots + [a_k, b_k]x_k + [a_{k+1}, b_{k+1}]}{[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}]},$$

subject to $A_1x_1 + \dots + A_kx_k \leq b,$

$$x_1 \geq 0, \dots, x_k \geq 0, \tag{1}$$

where A_i for $i = 1, \dots, k$ and b are m -dimensional constant column vectors. It is also assumed that $[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}] > 0, \forall x^T = (x_1, \dots, x_k) \in X$, where X is compact and non empty feasible region of problem (1). Setting variable,

$$z = \frac{1}{[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}]},$$

and using convex combination of each interval yields the following linear programming problem with an additional variable and two additional constraints as:

$$\text{Minimize } a_1y_1 + \dots + a_ky_k + a_{k+1}z$$

subject to $c_1y_1 + \dots + c_ky_k + c_{k+1}z \leq 1,$

$$d_1y_1 + \dots + d_ky_k + d_{k+1}z \geq 1,$$

$$A_1y_1 + \dots + A_ky_k \leq bz,$$

$$y_1 \geq 0, \dots, y_k \geq 0, z \geq 0. \tag{2}$$

If $(y_1^*, \dots, y_k^*, z^*)$ be the optimal solution for problem (2) then the optimal solution of problem (1) is $(\frac{y_1^*}{z^*}, \dots, \frac{y_k^*}{z^*})$.

FORMULATION OF THE PROBLEM

The extended general form of a two-level LFP problem with interval coefficients in the objective functions is as follows:

$$\text{Minimize } z_2(x_1, \dots, x_k) = \frac{[a_1, b_1]x_1 + \dots + [a_k, b_k]x_k + [a_{k+1}, b_{k+1}]}{[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}]}$$

$$\text{Minimize } z_2(x_1, \dots, x_k) = \frac{[e_1, f_1]x_1 + \dots + [e_k, f_k]x_k + [e_{k+1}, f_{k+1}]}{[g_1, h_1]x_1 + \dots + [g_k, h_k]x_k + [g_{k+1}, h_{k+1}]},$$

subject to $A_1x_1 + \dots + A_kx_k \leq b,$

$$x_1 \geq 0, \dots, x_k \geq 0. \tag{3}$$

We assumed that the denominators of functions $z_1(x_1, \dots, x_k)$ and $z_2(x_1, \dots, x_k)$ are positive for all $(x_1, \dots,$

$x_k) \in X$, where X is non empty polyhedral set of feasible points. Also for each value of the first-level variable (x_1, \dots, x_l) , there will be a unique solution to the second-level problem (x_{l+1}, \dots, x_k) .

To apply the idea of the K th best method, we must first show that the solution of problem (3) occurs at an extreme point.

Lemma 1: The linear fractional function with interval coefficients and positive denominator for all feasible points is a quasi-concave function.

Proof:

The compact general shape of a linear fractional function with interval coefficients is as follows:

$$f(x) = \frac{[A, B]x + [C, D]}{[E, F]x + [G, H]}, \tag{4}$$

and moreover, $[E, F]x + [G, H] > 0$, for all feasible points x .

To prove this, it suffices to show that the set $S = \{x \mid f(x) \geq \alpha\}$ is a convex set.

Let x_1 and $x_2 \in S$, it must be proven that $\forall \beta \in [0, 1]$: $\beta x_1 + (1 - \beta)x_2 \in S$, or equivalently $f(\beta x_1 + (1 - \beta)x_2) > \alpha$.

The above inequality can be written equivalently as follows:

$$\frac{[A, B](\beta x_1 + (1 - \beta)x_2) + [C, D]}{[E, F](\beta x_1 + (1 - \beta)x_2) + [G, H]} \geq \alpha. \tag{5}$$

From the fact that x_1 and $x_2 \in S$, two inequalities below are obtained,

$$[A, B]x_1 + [C, D] \geq \alpha ([E, F]x_1 + [G, H]), \text{ and} \tag{6}$$

$$[A, B]x_2 + [C, D] \geq \alpha ([E, F]x_2 + [G, H]). \tag{7}$$

Multiplying (6) and (7) with β and $(1 - \beta)$, respectively and adding them yields the following inequality:

$$[A, B](\beta x_1 + (1 - \beta)x_2) + [C, D] \geq \alpha ([E, F](\beta x_1 + (1 - \beta)x_2) + [G, H]). \tag{8}$$

According to Calvete and Gale (1998), when the objective functions of the first and second levels, z_1 and z_2 , are quasi-concave and continuous functions and common constraint region to both levels is a non empty and compact polyhedral set, the inducible region of the two-level programming problem is comprised of the union of connected faces of the polyhedral. Bearing this in mind, Calvete and Gale (1998) proved that there is an extreme point of the polyhedral that solves the problem.

Theorem 1:

(a) The inducible region of problem (3) is formed by the union of connected faces of S .

(b) An optimal solution to problem (3) occurs at an extreme point of polyhedral S .

Proof: From Lemma 1, both objective functions z_1 and z_2 are quasi-concave, therefore, part (a) of Theorem 1 follows from Lemmas 2.1, 2.2 and 2.5 and part (b) follows from Theorem 2.1 (Calvete & Gale 1998).

FINDING A STACKELBERG SOLUTION

According to Theorem 1, it is concluded that the optimal solution of problem (3) occurs at an extreme point. Therefore, the K th best method can be applied to find the Stackelberg solution.

First approach:

Let $(x_1^1, \dots, x_l^1, \dots, x_k^1), \dots, (x_1^N, \dots, x_l^N, \dots, x_k^N)$ denote the N ordered basic feasible solutions of problem (3) such that:

$$z_1(x_1^i, \dots, x_l^i, \dots, x_k^i) \leq z_1(x_1^{i+1}, \dots, x_l^{i+1}, \dots, x_k^{i+1})$$

for $i = 1, \dots, N - 1$.

For a given $(x_1^i, \dots, x_l^i, \dots, x_k^i)$, follower's problem is formulated as:

$$\text{Minimize } z_2(x_{l+1}, \dots, x_k) = \frac{[e, f]x' + \dots + [e, f]x'_l + [e, f]x_{l+1} + \dots + [e, f]x_k + [e, f]x_{k+1}}{[g, h]x' + \dots + [g, h]x'_l + [g, h]x_{l+1} + \dots + [g, h]x_k + [g, h]x_{k+1}}$$

$$\text{subject to } A_l x_{l+1} + \dots + A_k x_k \leq b - A_l x_1^i - \dots - A_l x_l^i, \tag{9}$$

$$x_{l+1} \geq 0, \dots, x_k \geq 0.$$

Problem (9) is reduced to the equivalent following problem:

$$\text{Minimize } z_2(x_{l+1}, \dots, x_k) = \frac{[e, x'_1 + \dots + e, x'_l, f, x'_1 + \dots + f, x'_l] + [e, \dots, f, x_{l+1} + \dots + e, f, x_k + [e, \dots, f, x_{k+1}]}{[g, x'_1 + \dots + g, x'_l, h, x'_1 + \dots + h, x'_l] + [g, \dots, h, x_{l+1} + \dots + [g, h, x_k + [g, \dots, h, x_{k+1}]}$$

$$\text{subject to } A_l x_{l+1} + \dots + A_k x_k \leq b - A_l x_1^i - \dots - A_l x_l^i, \tag{10}$$

$$x_{l+1} \geq 0, \dots, x_k \geq 0.$$

Since problem (10) is an LFP with interval coefficients in the objective function, it can be equivalently transformed into problem (2) to achieve the optimal solution. Therefore, if the optimal solution is $(\tilde{x}_{l+1}^i, \dots, \tilde{x}_k^i)$, then finding a Stackelberg solution is equivalent to seeking the minimal index i such that:

$$(\tilde{x}_{l+1}^i, \dots, \tilde{x}_k^i) = (x_{l+1}^i, \dots, x_k^i).$$

Second approach:

To obtain a Stackelberg solution of the leader's problem in which the objective function of the follower is eliminated

from problem (3), we first need to solve the following problem to achieve the order of the extreme points:

$$\begin{aligned} \text{Minimize}_{x_1, \dots, x_k} z_1(x_1, \dots, x_k) &= \frac{[a_1, b_1]x_1 + \dots + [a_k, b_k]x_k + [a_{k+1}, b_{k+1}]}{[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}]}, \\ \text{subject to } A_1x_1 + \dots + A_kx_k &\leq b, \\ x_1 \geq 0, \dots, x_k &\geq 0. \end{aligned} \tag{11}$$

Problem (11) is transformed into the following equivalent linear programming problem:

$$\begin{aligned} \text{Minimize } a_1y_1 + \dots + a_ky_k + a_{k+1}z, \\ \text{subject to } c_1y_1 + \dots + c_ky_k + c_{k+1}z &\leq 1, \\ d_1y_1 + \dots + d_ky_k + d_{k+1}z &\geq 1, \\ A_1y_1 + \dots + A_ky_k - bz &\leq 0, \\ y_1 \geq 0, \dots, y_k \geq 0, z &\geq 0. \end{aligned} \tag{12}$$

Let $(y_1^1, \dots, y_k^1, z^1), \dots, (y_1^N, \dots, y_k^N, z^N)$ denote the N ordered basic feasible solutions to problem (12) such that:

$$\begin{aligned} a_1y_1^i + \dots + a_ky_k^i + a_{k+1}z^i &\leq a_1y_1^{i+1} + \dots + a_ky_k^{i+1} + a_{k+1}z^{i+1}, \\ \text{for } i &= 1, \dots, N-1. \end{aligned}$$

Similar to the above, the follower's problem in which the objective function of a leader is eliminated from problem (3) as follows:

$$\begin{aligned} \text{Minimize}_{x_1, \dots, x_k} z_2(x_1, \dots, x_k) &= \frac{[e_1, f_1]x_1 + \dots + [e_k, f_k]x_k + [e_{k+1}, f_{k+1}]}{[g_1, h_1]x_1 + \dots + [g_k, h_k]x_k + [g_{k+1}, h_{k+1}]}, \\ \text{subject to } A_1x_1 + \dots + A_kx_k &\leq b, \\ x_1 \geq 0, \dots, x_k &\geq 0. \end{aligned} \tag{13}$$

The above problem has to be solved after selecting an extreme point of the feasible region of the problem (12). The equivalent linear programming shape of problem (13) is as follows:

$$\begin{aligned} \text{Minimize } e_1y_1 + \dots + e_ky_k + e_{k+1}z, \\ \text{subject to } g_1y_1 + \dots + g_ky_k + g_{k+1}z &\leq 1, \\ h_1y_1 + \dots + h_ky_k + h_{k+1}z &\geq 1, \\ A_1y_1 + \dots + A_ky_k - bz &\leq 0, \\ y_1 \geq 0, \dots, y_k \geq 0, z &\geq 0. \end{aligned} \tag{14}$$

Now, for a selected extreme point $(y_1^i, \dots, y_k^i, z^i)$ of problem (12), problem (14) is formulated as follows to check for a Stackelberg solution:

$$\begin{aligned} \text{Minimize } e_{l+1}y_{l+1} + \dots + e_ky_k, \\ \text{subject to } g_{l+1}y_{l+1} + \dots + g_ky_k &\leq 1 - g_1y_1^i - \dots - g_ly_l^i - g_{k+1}z^i, \end{aligned}$$

$$\begin{aligned} h_ly_l + \dots + h_ky_k &\geq 1 - h_1y_1^i - \dots - h_ly_l^i - h_{k+1}z^i, \\ A_{l+1}y_{l+1} + \dots + A_ky_k &\leq bz^i - A_1x_1^i - \dots - A_ly_l^i, \\ y_l \geq 0, \dots, y_k &\geq 0. \end{aligned} \tag{15}$$

Let $(\tilde{y}_{l+1}^i, \dots, \tilde{y}_k^i)$ denote the optimal solution of problem (15). Then finding the Stackelberg solution is equivalent to seeking the minimal index i such that:

$$(\tilde{y}_{l+1}^i, \dots, \tilde{y}_k^i) = (y_{l+1}^i, \dots, y_k^i).$$

NUMERICAL EXAMPLES

This section demonstrates the two approaches which are described in the previous section and numerical examples are given:

$$\begin{aligned} \text{Minimize}_{x_1} z_1 &= \frac{[-4, -2]x_1 [3, 10]x_2 + [-3, -1]}{[2, 5]x_1 + [3, 6]x_2 + [1, 6]}, \\ \text{Minimize}_{x_2} z_2 &= \frac{[8, 10]x_1 [-3, -2]x_2 + [-2, -1]}{[1, 5]x_1 + [1, 3]x_2 + [1, 7]}, \end{aligned}$$

$$\begin{aligned} \text{subject to } -x_1 + 2x_2 &\leq 13, \\ 2x_1 + 3x_2 &\leq 37, \\ 2x_1 - x_2 &\leq 17, \\ 2x_1 - 3x_2 &\leq 11, \\ x_1 + 4x_2 &\geq 11, \\ 5x_1 + 2x_2 &\geq 19, \\ x_1 \geq 0, x_2 &\geq 0. \end{aligned} \tag{16}$$

Using the first approach:

For the feasible extreme points of the problem (16), this relation holds:

$$z_1(7, 1) \leq z_1(10, 3) \leq z_1(11, 5) \leq z_1(3, 2) \leq z_1(5, 9) \leq z_1 \tag{17}$$

We start by choosing point (7,1) and then the follower's problem becomes,

$$\begin{aligned} \text{Minimize}_{x_2} z_2 &= \frac{[-3, -2]x_2 + [54, 669]}{[1, 3]x_2 + [8, 42]}, \\ \text{subject to } 2x_2 &\leq 20, \\ 3x_2 &\leq 23, \\ -x_2 &\leq 3, \\ -3x_2 &\leq -3, \\ 4x_2 &\geq 4, \\ 2x_2 &\geq -16, \\ x_2 &\geq 0. \end{aligned} \tag{17}$$

Problem (17) is transformed into the following equivalent linear programming problem:

$$\begin{aligned}
 & \text{Minimize } -3y_2 + 54z, \\
 & \text{subject to } y_2 + 8z \leq 1, \\
 & 3y_2 + 42z \geq 1, \\
 & 2y_2 - 20z \leq 0, \\
 & 3y_2 - 23z \leq 0, \\
 & -y_2 - 3z \leq 0, \\
 & -3y_2 + 3z \leq 0, \\
 & 4y_2 - 4z \geq 0, \\
 & 2y_2 + 16z \geq 0, \\
 & y_2 \geq 0, z \geq 0.
 \end{aligned} \tag{18}$$

The optimal solution of problem (18) is: $\tilde{y}_2^1 = 0.1179$ and $\tilde{z}^1 = 0.0154$.

According to: $\tilde{x}_2^1 = \frac{\tilde{y}_2^1}{\tilde{z}^1} = 7.658 \neq 1 = x_2^1$, it is concluded that point (7,1) is not a Stackelberg solution. If the procedure similar to the above problem is continued, then point (5,9) is found to be a Stackelberg solution.

Using the second approach:

Problem (12) for this numerical example can be formulated as:

$$\begin{aligned}
 & \text{Minimize } f_1(y_1, y_2, z) = -4y_1 + 3y_2 - 3z \\
 & \text{subject to } 2y_1 + 3y_2 + z \leq 1, \\
 & 5y_1 + 6y_2 + 6z \geq 1, \\
 & -y_1 + 2y_2 - 13z \leq 0, \\
 & 2y_1 + 3y_2 - 37z \leq 0, \\
 & 2y_1 - y_2 - 17z \leq 0, \\
 & 2y_1 - 3y_2 - 11z \leq 0, \\
 & y_1 + 4y_2 - 11z \geq 0, \\
 & 5y_1 + 2y_2 - 19z \geq 0, \\
 & y_1 \geq 0, y_2 \geq 0, z \geq 0.
 \end{aligned} \tag{19}$$

For the feasible extreme points of problem (19) the following relation holds:

$$\begin{aligned}
 & f_1(0.3889, 0.0556, 0.0556) < f_1(0.2308, 0.1538, 0.0769) \\
 & < f_1(0.1351, 0.0405, 0.0135) < f_1(0.1209, 0.0549, 0.0110) \\
 & < f_1(0.1316, 0.2368, 0.0263) < f_1(0.0455, 0.3182, 0.0455).
 \end{aligned}$$

At the beginning, point $(y_1^1, y_2^1, z^1) = (0.3889, 0.0556, 0.0556)$ is selected to confirm a Stackelberg solution. Problem (16) can be formulated for this point as follows:

$$\begin{aligned}
 & \text{Minimize } -3y_2 \\
 & \text{subject to } y_2 \leq 0.5555, \\
 & 3y_2 \geq -1.3337, \\
 & 2y_2 \leq 1.1117, \\
 & 3y_2 \leq 1.2794, \\
 & -y_2 \leq 0.1674, \\
 & -y_2 \leq -0.1662, \\
 & 4y_2 \geq 0.2227, \\
 & 2y_2 \geq -0.8881, \\
 & y_2 \geq 0.
 \end{aligned}$$

The optimal solution for the above problem is $\tilde{y}_2^1 = 0.4265 \neq y_2^1 = 0.0556$, which suggests that the point $(y_1^1, y_2^1, z^1) = (0.3889, 0.0556, 0.0556)$ is not a Stackelberg solution. If we execute further this procedure, it is found that the point (0.1316, 0.2368, 0.0263) is an optimal solution. This point can be transformed into point (5,9) by using the relation $\frac{y_i}{z} = x_i$. Therefore, it is shown that the two approaches produced the same desired Stackelberg solution.

CONCLUSION

In this paper, two different approaches were introduced to obtain Stackelberg solutions to the two-level LFP problems with interval coefficients in the objective functions. In the first approach, the LFP with interval coefficients in the objective function and related linear programming are used. Meanwhile, in the second approach, only the linear programming problems were utilized. We show that these two different approaches produced the same Stackelberg solution.

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